MATH 579: Combinatorics

Exam 3 Solutions

1. Use a characteristic equation (not generating functions) to solve the following recurrence. $a_0 = 0, a_1 = 9$, $a_n = -6a_{n-1} - 9a_{n-2}$ $(n \geq 2)$.

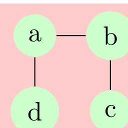
Our characteristic polynomial is $x^2 = -6x - 9$, which factors as $(x+3)^2 = 0$. Hence our general solution is $a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$. We now apply our initial conditions to get $0 = a_0 = \alpha_1(-3)^0 + \alpha_2 \cdot 0 \cdot (-3)^0 = \alpha_1$ and $9 = a_1 = \alpha_1(-3)^1 + \alpha_2 \cdot 1 \cdot (-3)^1 = -3\alpha_2$. This has solution $\alpha_1 = 0, \alpha_2 = -3$. Hence our solution is $a_n = 0(-3)^n + (-3)n(-3)^n = n(-3)^{n+1}$.

2. Use generating functions to solve the following recurrence.
$a_0 = 0, a_1 = 9$, $a_n = -6a_{n-1} - 9a_{n-2}$ $(n \geq 2)$.

Set $A(x) = \sum_{n\geq 0} a_n x^n$, multiply our relation by x^n and sum over $n \geq 2$. We get $\sum_{n\geq 2} a_n x^n = \sum_{n\geq 2} -6a_{n-1}x^n + \sum_{n\geq 2} -9a_{n-2}x^n = -6x\sum_{n\geq 2} a_{n-1}x^{n-1} - 9x^2 \sum_{n\geq 2} a_{n-2}x^{n-2}$. Hence $A(x) - a_0 - a_1 x = -6x(A(x) - a_0) - 9x^2 A(x)$, which rearranges to $A(x)(1+6x+9x^2) = 9x$, so $A(x) = \frac{9x}{1+6x+9x^2}$ is our generating function.

Version 1: There is no need for partial fractions, as $A(x) = \frac{9x}{(1+3x)^2} = (-3)\frac{(-3x)}{(1-(-3x))^2}$ is already in our dictionary. We have $A(x) = (-3)\sum_{n\geq 0} n(-3x)^n = (-3)\sum_{n\geq 0} n(-3)^n x^n$. Hence $a_n = (-3)n(-3)^n = n(-3)^{n+1}$.

Version 2: Lovers of partial fractions can write $A(x) = \frac{\alpha}{1+3x} + \frac{\beta}{(1+3x)^2}$, so $\alpha(1+3x)+\beta = 9x$. Equating coefficients, we get $\alpha = 3, \beta = -3$. So $A(x) = 3\sum_{n\geq 0}(-3)^n x^n - 3\sum_{n\geq 0}(n+1)(-3)^n x^n = \sum_{n\geq 0}(3 - 3(n+1))(-3)^n x^n = \sum_{n\geq 0} -3n(-3)^n x^n$. So, $a_n = -3n(-3)^n = n(-3)^{n+1}$.

3. Use inclusion/exclusion to find the chromatic polynomial for: 

We have $S = \{ab, bc, ad\}$, so $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(ab) - f_\geq(bc) - f_\geq(ad) + f_\geq(abc) + f_\geq(abd) + f_\geq(bc, ad) - f_\geq(abcd) = x^4 - 3x^3 + 3x^2 - x$.

4. Solve the following recurrence however you like: $a_0 = 0, a_n = 3a_{n-1} + 2^n + 3^n$ $(n \geq 1)$.

Version 1: The homogeneous version is easy: $a_n = 3a_{n-1}$, with general solution $a_n = A3^n$. The tricky bit is guessing a solution to the nonhomogeneous version. Since 3^n is in the general solution space, we instead multiply by n, guessing $a_n = j2^n + kn3^n$. Plugging in, we have $j2^n + kn3^n = 3(j2^{n-1} + k(n-1)3^{n-1}) + 2^n + 3^n$, which rearranges as $2^{n-1}(2j - 3j - 2) + 3^{n-1}(3kn - 3k(n-1) - 3) = 0$. Hence we need $-j - 2 = 0$ and $3k - 3 = 0$, i.e. $j = -2, k = 1$. So our general solution is $a_n = A3^n - 2^{n+1} + n3^n$. Applying our initial condition gives $0 = a_0 = A3^0 - 2^1 + 03^0 = A - 2$. Hence $A = 2$, and our solution is $a_n = 2 \cdot 3^n - 2^{n+1} + n3^n = (n+2)3^n - 2^{n+1}$.

Version 2: Set $A(x) = \sum_{n\geq 0} a_n x^n$, multiply both sides by x^n, and sum over $n \geq 1$. We get $\sum_{n\geq 1} a_n x^n = 3\sum_{n\geq 1} a_{n-1}x^n + \sum_{n\geq 1} 2^n x^n + \sum_{n\geq 1} 3^n x^n$, and hence $A(x) - a_0 = 3xA(x) + \frac{1}{1-2x} - 1 + \frac{1}{1-3x} - 1$, or $A(x)(1-3x) = \frac{1}{1-2x} + \frac{1}{1-3x} - 2$. Dividing, we get $A(x) = \frac{1}{(1-2x)(1-3x)} + \frac{1}{(1-3x)^2} - \frac{2}{1-3x}$. We need a bit of partial fractions, $\frac{1}{(1-2x)(1-3x)} = \frac{\alpha}{1-2x} + \frac{\beta}{1-3x}$. Hence $1 = \alpha(1-3x) + \beta(1-2x)$, so $\alpha = -2, \beta = 3$. Hence $A(x) = \frac{-2}{1-2x} + \frac{3}{1-3x} + \frac{1}{(1-3x)^2} - \frac{2}{1-3x} = \frac{-2}{1-2x} + \frac{1}{(1-3x)^2} + \frac{1}{1-3x} = -2\sum_{n\geq 0} 2^n x^n + \sum_{n\geq 0}(n+1)3^n x^n + \sum_{n\geq 0} 3^n x^n = \sum_{n\geq 0}((-2)\cdot 2^n + (n+1)3^n + 3^n)x^n$. Hence $a_n = -2 \cdot 2^n + (n+1)3^n + 3^n = -2^{n+1} + (n+2)3^n$.

5. Count the number of solutions to $a + b + c + d = n$ in nonnegative integers a, b, c, d, such that a is a multiple of 4, b is at most 1, and d is either 0 or 2.

The number of solutions is counted by the generating function $(1+x^4+x^8+\cdots)(1+x)(1+x+x^2+x^3+\cdots)(1+x^2) = \frac{1}{1-x^4}(1+x)\frac{1}{1-x}(1+x^2) = \frac{(1+x)(1+x^2)}{(1+x^2)(1-x^2)(1-x)} = \frac{1+x}{(1+x)(1-x)(1-x)} = \frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n$. Hence the desired solution is $n+1$.

6. Count the number of solutions to $a + b + c + d = 30$ in nonnegative integers a, b, c, d, such that $a \leq 9, b \leq 9, c \leq 9, d \leq 14$.

We have $S = \{s_a, s_b, s_c, s_d\}$, where s_a means $a \geq 10$, s_b means $b \geq 10$, s_c means $c \geq 10$, and s_d means $d \geq 15$. We have a lot of symmetry, with d going its own way. So, $f_=(\emptyset) = f_\geq(\emptyset) - 3f_\geq(s_a) - f_\geq(s_d) + 3f_\geq(s_a s_b) + 3f_\geq(s_a s_d) - f_\geq(s_a s_b s_c) - 3f_\geq(s_a s_b s_d) + f_\geq(s_a s_b s_c s_d) = \left(\!\binom{4}{30}\!\right) - 3\left(\!\binom{4}{20}\!\right) - \left(\!\binom{4}{15}\!\right) + 3\left(\!\binom{4}{10}\!\right) + 3\left(\!\binom{4}{5}\!\right) - \left(\!\binom{4}{0}\!\right) - 3 \cdot 0 + 0 = 352$.